only the value $z(t)$ of the phase variable z). As a rule this situation is common in linear differential games [8, 9].

REFERENCES

1. Pontriagin, L. S., On the theory of differential games. Uspekhi Matem. Nauk, Vol. 21, №4, 1966.
2. Filippov, A. F., On certain aspects of the theory of optimal control. Vestn. MGU, Ser. Matem., Mekhan., Astronom., Fiz., Khim., №2, 1959.
3. Gabasov, R. and Kirillova, F. M., Qualitative Theory of Optimal Processes. Moscow, "Nauka", 1971.
4. Boltianskii, V. G., Mathematical Methods of Optimal Control. Moscow, "Nauka", 1969.
5. Natanson, I. P., Theory of Functions of a Real Variable. Moscow, Gostekhteoretizdat, 1957.
6. Satimov, N., On the smoothness of the Bellman functions for linear systems. Dokl. Akad. Nauk Uzbek SSR, №7, 1971.
7. Krasovskii, N. N., Game Problems on the Contact of Motions. Moscow, "Nauka", 1970.
8. Krasovskii, N. N. and Subbotin, A. I., A differential game of guidance. Differentsial'nye Uravneniia, Vol. 6, №4, 1970.
9. Batukhtin, V. D. and Subbotin, A. I., On conditions of completing a pursuit game. Izv. Akad. Nauk SSSR, Tekhnicheskaia Kibernetika, №1, 1972.

Translated by N. H. C.

UDC 62-50

ON THE ESTIMATION OF CERTAIN PERFORMANCE INDICES IN LINEAR STATIONARY CONTROLLED SYSTEMS

PMM Vol. 38, №1, 1974, pp. 45-48
V. G. TREIVAS
(Moscow)
(Received August 16, 1972)

We consider the behavior of a closed-loop stationary controlled system when the forcing functions belong to a certain class of functions (the Bulgakov problem [1, 2]). We derive estimates for the modulus of the maximum value of the output and for the largest accumulation of system errors.

1. Consider the system of equations

$$c_0 y^{(n)} + c_1 y^{(n-1)} + \ldots + c_{n-2} y'' + y' = k\varepsilon_x(t) \qquad (1.1)$$
$$y^{(n-1)}(0) = \ldots = y(0) = 0$$
$$\varepsilon_x(t) = x(t) - y(t)$$

Equations (1.1) describe the behavior of a closed-loop linear astatic automatic control system with rigid negative feedback, widely prevalent in practice, in which $x(t)$ is the forcing function, $y(t)$ is the quantity being controlled, $\varepsilon_x(t)$ is the system error. The forcing function is not known in advance and belongs to a class F of piecewise-continuous functions satisfying the condition

$$|x(t)| \leqslant m_1 \qquad (1.2)$$

The system's performance indices are the maximum value $y_{\max}(t)$ of the output under the worst functions $x(t)$ from class F, the largest accumulation $\varepsilon_{\max}(t)$ of system error, and its limit value ε_∞ as $t \to \infty$

$$\varepsilon_{\max}(t) = \max_{x \in F} \varepsilon_x(t), \qquad \varepsilon_\infty = \lim_{t\to\infty} \varepsilon_{\max}(t)$$

The stated problem is known as the Bulgakov problem [1, 2]. For higher-order systems considerable computational difficulties are connected with the exact solution and the investigation of the influence of the control system's parameters on $\varepsilon_{\max}(t)$ and ε_∞. There is thus a pressing need to obtain guaranteed estimates for these quantites. Theorem 1 gives an estimate on $|y_{\max}(t)|$ for forcing functions from the function class F

Theorem 1. Let the gain k be chosen such that all roots z_i of the characteristic polynomial of the closed-loop system (1.1) are located in the left halfplane and that the inequality

$$-\operatorname{Re} z_i > a_0, \quad a_0 > 0, \quad i = 1, 2, \ldots, n \qquad (1.3)$$

is satisfied. Then

$$|y_{\max}(t)| \leqslant \frac{m_1 k}{c_0 a_0^n}\left(1 - e^{-a_0 t}\sum_{i=0}^{n-1}\frac{(a_0 t)^i}{i!}\right) \qquad (1.4)$$

$$|y_{\max}(\infty)| \leqslant \frac{m_1 k}{c_0 a_0^n}$$

Proof. Taking the Laplace transform in Eqs. (1.1) we have

$$Y(p) = k\,[c_0 (p - z_1) \ldots (p - z_n)]^{-1} X(p)$$

Here z_i are roots of the characteristic polynomial $N(p)$ of the closed-loop system (1.1)

$$N(p) = p^n + \frac{c_1}{c_0} p^{n-1} + \ldots + \frac{1}{c_0} p + \frac{k}{c_0}$$

Having applied the convolution theorem and chosen the worst forcing function $x(t)$ from the class of piecewise-continuous modulus-bounded functions, we obtain

$$|y_{\max}(t)| = \frac{m_1 k}{c_0}\int_0^t |g(\tau)|\, d\tau \qquad (1.5)$$

Here $g(\tau)$ is the original corresponding to the transform $[(p - z_1) \ldots (p - z_n)]^{-1}$. The Laplace transform inversion theorem and the fulfillment of the hypotheses of Jordan's lemma allow to represent $g(\tau)$ as

$$g(\tau) = \sum_{i=1}^{n}\left[\prod_{j=1,\, i\neq j}^{n} (z_i - z_j)\right]^{-1} \exp(z_i \tau) \qquad (1.6)$$

The right hand side of expression (1.6) is a divided difference of order $n-1$ of the function $e^{p\tau}$ at points $z_1, \ldots, z_n$ of the complex plane; representation (1.6) remains true under any coincidences of the points $z_1, \ldots, z_n$ [3]. By virtue of estimate derived in [3], for the modulus of a divided difference, we have

$$|g(\tau)| \leqslant \frac{1}{(n-1)!} \max_{z_i} |(e^{p\tau})_p^{(n-1)}| \leqslant \frac{\tau^{n-1} e^{-a_0\tau}}{(n-1)!} \quad (1.7)$$

We obtain estimates (1.4) by substituting (1.7) into expression (1.5).

Note 1. The Laplace transform of the system error has the form

$$E(p) = (1 - k/c_0 N(p)) X(p)$$

Therefore, for forcing functions satisfying condition (1.3), the estimate

$$\varepsilon_\infty \leqslant m_1 + km_1/c_0$$

is valid for the maximum accumulation of error.

Note 2. The choice of the gain k and its relation to the degree of stability a_0 of the closed-loop system (1.1) have been examined in [4].

2. Let the forcing functions $x(t)$ belong to a class F_1 of piecewise-continuous functions with a bounded rate of variation

$$x(0) = 0, \qquad |\dot{x}(t)| \leqslant m$$

A lower bound for the quantity ε_∞ is derived in Theorem 2.

Theorem 2. Let the gain k be chosen such that system (1.1) is asymptotically stable. Then for forcing functions from class F_1

$$\varepsilon_\infty \geqslant m / k$$

Proof. We set $x_1 = mt$, then

$$\varepsilon_\infty \geqslant \lim_{t\to\infty} \varepsilon_{x_1}(t)$$

Since

$$E(p) = \frac{pL(p)}{pL(p)+k} X_1(p) \quad (2.1)$$

$$L(p) = c_0 p^{n-1} + c_1 p^{n-2} + \ldots + 1, \qquad X_1(p) = mp^{-2}$$

by the limit value theorem we obtain

$$\lim_{t\to\infty} \varepsilon_{x_1}(t) = \lim_{p\to 0} pE(p) = m/k$$

The same follows from the fact that the magnitude of the error in an astatic system in steady-state under the linear forcing being considered equals mk^{-1} [5].

An upper bound for the quantities $\varepsilon_{\max}(t)$ and ε_∞ is given by Theorem 3.

Theorem 3. Let the gain k be chosen such that all roots z_i of the characteristic polynomial $N(p)$ of closed-loop system (1.1) are located in the left halfplane and condition (1.3) is satisfied. Then for forcing functions from class F_1

$$\varepsilon_{\max}(t) \leqslant \frac{km}{c_0 a_0^{n+1}} \left(n - e^{-a_0 t} \sum_{i=0}^{n} \frac{(n-i)(a_0 t)^i}{i!} \right), \quad \varepsilon_\infty \leqslant \frac{kmn}{c_0 a_0^{n+1}} \quad (2.2)$$

Proof. From expression (2.1) for the error $E(p)$ we obtain

$$E(p) = \frac{L(p) X(p)}{pL(p) + k}$$

Hence by the convolution theorem

$$\varepsilon(t)=\int_0^t s(t-\tau)\,x'(\tau)\,d\tau$$

Here $s(t)$ is the original corresponding to the transform

$$L(p)\,(pL(p)+k)^{-1}$$

For forcing functions $x(t)$ from class F_1

$$\varepsilon_{\max}(t)=m\int_0^t |s(\tau)|\,d\tau \tag{2.3}$$

By the Laplace transform inversion theorem and Jordan's lemma we have

$$s(\tau)=\sum_{i=1}^{n}\Big[c_0\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)\Big]^{-1}L(z_i)\exp(z_i\tau) \tag{2.4}$$

Since z_i are the roots of the equation $N(p)=0$, then $z_i\,L(z_i)=-k$. Taking this into account, from (2.4) we obtain

$$s(\tau)=-\frac{k}{c_0}\sum_{i=1}^{n}\Big[\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i-z_j)\Big]^{-1}z_i^{-1}\exp(z_i\tau) \tag{2.5}$$

The right-hand side of expression (2.5) is a divided difference of order $n-1$ of the function $e^{p\tau}p^{-1}$ at the points $z_1,\ldots,z_n$ of the complex plane. The estimate

$$|s(\tau)|\leqslant\frac{k}{c_0(n-1)!}\max_{z_i}\left|\left(\frac{e^{p\tau}}{p}\right)_p^{(n-1)}\right| \tag{2.6}$$

is valid for $|s(\tau)|$. Differentiating (2.6) $n-1$ times and taking (1.3) into account, we obtain

$$|s(\tau)|\leqslant\frac{ke^{-a_0\tau}}{c_0(n-1)!}\sum_{i=1}^{n}C_{n-1}^{n-i}\frac{(i-1)!\,\tau^{n-i}}{a_0^{\,i}} \tag{2.7}$$

Substituting expression (2.7) into (2.3) and integrating, we obtain

$$\varepsilon_{\max}(t)\leqslant\frac{km}{c_0a_0^{n+1}}\left(n-e^{-a_0t}\sum_{i=1}^{n}\sum_{j=0}^{n-i}\frac{(a_0t)^j}{j!}\right) \tag{2.8}$$

We obtain estimate (2.2) by replacing the double sum in (2.8) by the sum

$$\sum_{i=0}^{n}\frac{(n-i)\,(a_0t)^i}{i!}$$

For $n=2$, if the roots z_1 and z_2 of the characteristic polynomial $N(p)$ of system (1.1) are real, then the lower bound coincides with the exact value of the error, while for multiple roots $z_1=z_2$ the upper bound coincides with the exact value of the error. In particular, the problem of the exactness of system (1.1) under the worst forcings was examined in [4] and other upper and lower estimates were obtained for the largest accumulation of error.

REFERENCES

1. Bulgakov, B. V., On the accumulation of perturbations in linear oscillatory systems with constant parameters. Dokl. Akad. Nauk SSSR, Vol. 51, № 5, 1946.

2. Bulgakov, B. V. and Kuzovkov, N. T., On the accumulation of disturbances in linear systems with variable parameters. PMM Vol. 14, №1, 1950.
3. Gel'fond, A. O., Finite Difference Calculus. Moscow, "Nauka", 1967.
4. Gnoenskii, L. S., On the connection between certain performance indices in linear stationary controlled systems. Dokl. Akad. Nauk SSSR, Vol. 181, №1, 1968.
5. Gnoenskii, L. S., Kamenskii, G. A. and El'sgol'ts, L. E., Mathematical Foundations of the Theory of Controlled Systems. Moscow, "Nauka", 1969.

Translated by N. H. C.

UDC 531.36

NUMERICAL STABILITY INVESTIGATION OF THE LAGRANGE SOLUTIONS OF AN ELLIPTIC RESTRICTED THREE-BODY PROBLEM

PMM Vol. 38, №1, 1974, pp. 49-55

A. P. MARKEEV and A. G. SOKOL'SKII
(Moscow)

(Received June 8, 1973)

We investigate numerically the triangular points of solutions of the elliptic restricted three-body problem. For the planar problem we have constructed, in the space of parameters e and μ (e is the eccentricity, μ is the ratio of the mass of the smaller of the two main bodies to the sum of their masses), the stability region for a majority of initial conditions and the region of formal stability. For resonant values of the parameters we found Liapunov-instability or stability in the fourth approximation relative to the coordinates and momenta of the perturbed motion. For spatial problems we have obtained a statement of stability in the fourth approximation.

1. We examine the motion of three material points attracted to each other by Newton's law. The differential equations of motion of the three-body problem allow a particular solution, corresponding to the motion under which the three bodies form an equilateral triangle rotating in their own plane around the center of mass of the three-body system. We investigate the stability of this particular solution for the case of the elliptic restricted three-body problem.

We consider the planar problem. We select the measurement units such that the distance between the bodies of finite mass, the sum of their masses, and the gravitational constant equal unity. Then in Nechvile coordinates with true anomaly ν as the independent variable, the expansion of the Hamiltonian function of the perturbed motion has the form

$$H = H_2 + H_3 + H_4 + \ldots \tag{1.1}$$

$$H_2 = \frac{1}{2}(p_1^2 + p_2^2) + q_2 p_1 - q_1 p_2 + \frac{1}{1 + e\cos\nu}\left(\frac{1}{8} q_1^2 - \frac{5}{8} q_2^2 - kq_1 q_2\right) + \frac{e\cos\nu}{2(1 + e\cos\nu)}(q_1^2 + q_2^2)$$

$$H_3 = \frac{1}{1 + e\cos\nu}\left(-\frac{7\sqrt{3}k}{36} q_1^3 + \frac{3\sqrt{3}}{16} q_1^2 q_2 + \frac{11\sqrt{3}k}{12} q_1 q_2^2 + \frac{3\sqrt{3}}{16} q_2^3\right)$$